THE FIXED SUBGROUPS OF HOMEOMORPHISMS OF SEIFERT MANIFOLDS

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Abstract. Let M be a compact connected orientable Seifert manifold with hyperbolic orbifold B_M, and $f_\pi : \pi_1(M) \to \pi_1(M)$ be an automorphism induced by an orientation-reversing homeomorphism f of M. We give a bound on the rank of the fixed subgroup of f_π, namely, $\mathrm{rankFix}(f_\pi) < 2\mathrm{rank}\pi_1(M)$, which is similar to the inequalities on surface groups and hyperbolic 3-manifold groups.

1. Introduction

For a group G and an endomorphism $\phi : G \to G$, the *fixed subgroup* of ϕ is

$$\mathrm{Fix}(\phi) := \{g \in G | \phi(g) = g\},$$

which is a subgroup of G. Let $\mathrm{rank}G$ denote the *rank* of G, which means the minimal number of the generators of G.

For a free group F and an automorphism ϕ, M. Bestvina and M. Handel [BH] solved the Scott conjecture:

Theorem 1.1 (Bestvina-Handel). *Let F be a free group and ϕ be an automorphism of F. Then*

$$\mathrm{rankFix}(\phi) \leq \mathrm{rank}F.$$

In the paper by B. Jiang, S. Wang and Q. Zhang [JWZ], it is proved that

Theorem 1.2 (Jiang-Wang-Zhang). *Let S be a compact surface and ϕ be an endomorphism of $\pi_1(S)$. Then*

$$\mathrm{rankFix}(\phi) \leq \mathrm{rank}\pi_1(S).$$

In a recent paper [WZ], J. Wu and Q. Zhang generalized Theorem 1.2 to a family endomorphisms of a surface group. In [LW] (see [Z2] for an enhance version), J. Lin and S. Wang showed that

Theorem 1.3 (Lin-Wang). *Let M be a compact orientable hyperbolic 3-manifold with finite volume and ϕ be an automorphism of $\pi_1(M)$. Then*

$$\mathrm{rankFix}(\phi) < 2\mathrm{rank}\pi_1(M).$$

In this paper, we consider the fixed subgroups of automorphisms of Seifert 3-manifold groups.

Suppose M is a compact orientable 3-manifold. We say that M is a *Seifert manifold*, if M possesses a *Seifert fibration* which is a decomposition of M into disjoint simple closed curves, called *fibers*, such that each fiber has a solid torus neighborhood consisting of a union of fibers. Identifying each fiber of M to a point, we get a set B_M, called the *orbifold* of M, which has a natural 2-orbifold structure with singular points consisting of cone points. It is useful to think of a Seifert manifold as a circle bundle over a 2-orbifold. For brevity, we say an orbifold it means a compact 2-orbifold with singular points consisting of cone points in the following. An orbifold (or

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surface) is called *hyperbolic* if it has negative Euler characteristics. A hyperbolic orbifold is orbifold covered by a hyperbolic surface and admits a hyperbolic structure with totally geodesic boundary. For more information about orbifolds, see [JWW, §1 and §2] and [S, §2]. A map f on a Seifert manifold M is called *fiber-preserving* if it maps fibers to fibers. If f is fiber-preserving, then it induces a map $f' : B_M \to B_M$ on the orbifold B_M.

In the following, all spaces are assumed to be connected and compact unless it is specially stated otherwise. For a set X, let $\#X$ denote the number of points in X.

The main result of this paper is

Theorem 1.4. *Suppose M is a compact connected orientable Seifert manifold (closed or with toroidal boundary) with hyperbolic orbifold B_M, and $f_\pi : \pi_1(M) \to \pi_1(M)$ is an automorphism induced by an orientation-reversing homeomorphism $f : M \to M$. Then*

$$\mathrm{rankFix}(f_\pi) < 2\mathrm{rank}\pi_1(M).$$

Remark 1.5. By the well known Geometrization Theorem, M is a Seifert manifold with hyperbolic orbifold B_M if and only if M admits a geometric structure based on one of the two geometries : $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL(2,\mathbb{R})}$. The condition that f is orientation-reversing is necessary. If f is orientation-preserving, then the fixed subgroup $\mathrm{Fix}(f_\pi)$ can be infinitely generated, see [Z1, Example 5.2].

Remark 1.6. By [Z1, Example 5.1], one can show that there is an orientation-reversing homeomorphism f of $M_n = S_n \times S^1$, where S_n is a closed orientable surface of genus $n \geq 2$, such that $\mathrm{rankFix}(f_\pi) = 4n - 2$. Therefore, for any $\varepsilon > 0$, there exists a Seifert manifold M_n and an orientation-reversing homeomorphism f of M_n, such that

$$\frac{\mathrm{rankFix}(f_\pi)}{\mathrm{rank}\pi_1(M_n)} = \frac{4n-2}{2n+1} > 2 - \varepsilon.$$

Theorem 1.4 is inspired by the following proposition [Z1, Corollary 1.4].

Proposition 1.7. *Suppose $f : M \to M$ is a homeomorphism of a compact connected orientable Seifert manifold with hyperbolic orbifold B_M. Let $f_* : \pi_1(M, x) \to \pi_1(M, x)$ be the induced automorphism, where x is a fixed point contained in an essential fixed point class of f. Then*

$$\mathrm{rankFix}(f_*) < 2\mathrm{rank}\pi_1(M).$$

The paper is organized as follows. In Section 2, we will give some background on fixed points and fixed subgroups of a selfmap. In Section 3, we will give some useful facts on Fuchsian groups. In Section 4, we will consider the special case of Theorem 1.4 that M is an orientable circle bundle over an orientable hyperbolic surface. Finally, we will finish the proof of Theorem 1.4 in Section 5, and give some examples and questions in Section 6.

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2. Fixed points and fixed subgroups

Let X be a connected compact polyhedron and $f : X \to X$ be a selfmap. In this section, we introduce some facts on fixed point classes and fixed subgroups of f.

The fixed point set

$$\mathrm{Fix}f := \{x \in X | f(x) = x\}$$

splits into a disjoint union of *fixed point classes*: two fixed points are in the same class if and only if they can be joined by a *Nielsen path*, which is a path homotopic (rel. endpoints) to its own f-image. For each fixed point class $\mathbf{F}$, a homotopy invariant *index* $\mathrm{ind}(f, \mathbf{F}) \in \mathbb{Z}$ is defined. A fixed point class is *essential* if its index is non-zero, otherwise, called *inessential* (see [J] for an introduction).

Although there are several approaches to define fixed point classes, we state the one using paths and introduce another homotopy invariant *rank* $\mathrm{rank}(f, \mathbf{F}) \in \mathbb{Z}$ for each fixed point class $\mathbf{F}$ (see [JWZ, §2]), which plays a key role in this paper.

Definition 2.1. By an *f-route* we mean a homotopy class (rel. endpoints) of path $w : I \to X$ from a point $x \in X$ to $f(x)$. For brevity we shall often say the path w (in place of the path class $[w]$) is an f-route at $x = w(0)$. An f-route w gives rise to an endomorphism

$$f_w : \pi_1(X, x) \to \pi_1(X, x),\ [a] \mapsto [w(f \circ a)\overline{w}]$$

where a is any loop based at x, and $\overline{w}$ denotes the reverse of w. For brevity, we will write

$$f_\pi : \pi_1(X) \to \pi_1(X)$$

when w and the base point x are omitted.

Two f-routes $[w]$ and $[w']$ are *conjugate* if there is a path $q : I \to X$ from $x = w(0)$ to $x' = w'(0)$ such that $[w'] = [\overline{q}w(f \circ q)]$, that is w' and $\overline{q}w(f \circ q)$ homotopic rel. endpoints. We also say that the (possibly tightened) f-route $\overline{q}w(f \circ q)$ is obtained from w by an *f-route move* along the path q.

Note that a constant f-route w corresponds to a fixed point $x = w(0) = w(1)$ of f, and the endomorphism f_w becomes the usual

$$f_* : \pi_1(X, x) \to \pi_1(X, x),\ [a] \mapsto [f \circ a],$$

where a is any loop based at x. Two constant f-routes are conjugate if and only if the corresponding fixed points can be joint by a Nielsen path. This gives the following definition.

Definition 2.2. With an f-route w (more precisely, with its conjugacy class) we associate a *fixed point class* $\mathbf{F}_w$ of f, which consists of the fixed points that correspond to constant f-routes conjugate to w. Thus fixed point classes are associated bijectively with conjugacy classes of f-routes. A fixed point class $\mathbf{F}_w$ can be empty if there is no constant f-route conjugate to w. Empty fixed point classes are inessential and distinguished by their associated route conjugacy classes.

Definition 2.3. The *fixed subgroup* of the endomorphism f_w is the subgroup

$$\mathrm{Fix}(f_w) := \{\gamma \in \pi_1(X, w(0)) | f_w(\gamma) = \gamma\}.$$

The *stabilizer* of the fixed point class $\mathbf{F}_w$ is defined to be

$$\mathrm{Stab}(f, \mathbf{F}_w) := \mathrm{Fix}(f_w),$$

it is well defined up to isomorphism because conjugate f-routes have isomorphic stabilizers. Hence, we have the *rank* of $\mathbf{F}_w$ defined as

$$\mathrm{rank}(f, \mathbf{F}_w) := \mathrm{rank}\mathrm{Stab}(f, \mathbf{F}_w) = \mathrm{rank}\mathrm{Fix}(f_w).$$

The following are some facts on stabilizer (see [JWZ, §2]).

Fact (Homotopy invariance). A homotopy $H = \{h_t\}_{t\in I} : X \to X$ gives rise to a bijective correspondence $H : \mathbf{F}_{w_0} \mapsto \mathbf{F}_{w_1}$ from h_0-fixed point classes to h_1-fixed point classes, and

$$\mathrm{ind}(h_0, \mathbf{F}_{w_0}) = \mathrm{ind}(h_1, \mathbf{F}_{w_1}),\ \mathrm{Stab}(h_0, \mathbf{F}_{w_0}) \cong \mathrm{Stab}(h_1, \mathbf{F}_{w_1}),$$

which indicates that the index $\mathrm{ind}(f, \mathbf{F}_w)$ and the rank $\mathrm{rank}(f, \mathbf{F}_w)$ of a fixed point class are both homotopy invariants.

Fact (Morphism). A morphism from a selfmap $f : X \to X$ to a selfmap $g : Y \to Y$ means a map $h : X \to Y$ such that $h \circ f = g \circ h$. It induces a natural function $w \mapsto h \circ w$ from f-routes to g-routes and a function $\mathbf{F}_w \mapsto \mathbf{F}_{h\circ w}$ from f-fixed point classes to g-fixed point classes, such that

$$h(\mathbf{F}_w) \subseteq \mathbf{F}_{h\circ w},\ h_*\mathrm{Stab}(f, \mathbf{F}_w) \leq \mathrm{Stab}(g, \mathbf{F}_{h\circ w}).$$

The following are useful lemmas on covering spaces and fixed point classes.

Lemma 2.4. *Let $p : \widetilde{M} \to M$ be a finite covering of a compact manifold M, and $f : M \to M$ be a homeomorphism. Suppose $\tilde{f} : \widetilde{M} \to \widetilde{M}$ is a lifting of f, and the $\tilde{f}$-route $\tilde{w}$ is a lifting of the f-route w. Then the f-fixed point class $\mathbf{F}_w$ associated to w is essential if and only if the $\tilde{f}$-fixed point class $\mathbf{F}_{\tilde{w}}$ associated to $\tilde{w}$ is essential, moreover,*

$$\mathrm{ind}(\tilde{f}, \mathbf{F}_{\tilde{w}}) = n \times \mathrm{ind}(f, \mathbf{F}_w)$$

where n is a positive integer.

Proof. Since indices of fixed point classes are homotopy invariants of the map f, via perturbation, we may assume that all fixed points of f are isolated.

Firstly, we claim that $\mathbf{F}_{\tilde{w}}$ is nonempty if and only if $\mathbf{F}_w$ is nonempty, and $p(\mathbf{F}_{\tilde{w}}) = \mathbf{F}_w$.

If $\mathbf{F}_{\tilde{w}}$ is nonempty, then for any point $\tilde{x} \in \mathbf{F}_{\tilde{w}} \subseteq \mathrm{Fix}\tilde{f}$, the $\tilde{f}$-route $\tilde{w}$ is conjugate to $\tilde{x}$, namely, there is a path $\gamma : I \to \widetilde{M}$ from $\tilde{w}(0)$ to $\tilde{x}$ such that $\overline{\gamma}\tilde{w}(\tilde{f} \circ \gamma) \simeq \tilde{x}$ rel. $\tilde{x}$. Hence

$$(\overline{p \circ \gamma})w(f(p \circ \gamma)) = p(\overline{\gamma}\tilde{w}(\tilde{f} \circ \gamma)) \simeq p(\tilde{x}).$$

Namely, the f-route w is conjugate to the point $p(\tilde{x}) \in \mathbf{F}_w \subseteq \mathrm{Fix}f$. So $p(\mathbf{F}_{\tilde{w}}) \subseteq \mathbf{F}_w$.

If $\mathbf{F}_w$ is nonempty, then for any point $x \in \mathbf{F}_w \subseteq \mathrm{Fix}f$, the f-route w is conjugate to x, namely, there is a path c from $w(0)$ to x such that $\overline{c}w(f \circ c) \simeq x$ rel. x. Let $\tilde{c}$ be a lifting of c from $\tilde{w}(0)$ to a point $\tilde{x} \in p^{-1}(x)$. Then $\overline{\tilde{c}}\tilde{w}(\tilde{f} \circ \tilde{c})$ is a lifting of the contractible loop $\overline{c}w(f \circ c)$. Hence $\overline{\tilde{c}}\tilde{w}(\tilde{f} \circ \tilde{c})$ is also a contractible loop and $\tilde{f}(\tilde{x}) = \tilde{x}$, namely, the $\tilde{f}$-route $\tilde{w}$ is conjugate to the point $\tilde{x} \in \mathbf{F}_{\tilde{w}} \subseteq \mathrm{Fix}\tilde{f}$. So $\mathbf{F}_w \subseteq p(\mathbf{F}_{\tilde{w}})$ and the claim holds.

Secondly, we prove $\mathrm{ind}(\tilde{f}, \mathbf{F}_{\tilde{w}}) = n \times \mathrm{ind}(f, \mathbf{F}_w)$.

If $\mathbf{F}_w$ is empty, then $\mathbf{F}_{\tilde{w}}$ is also empty according to the claim above, and the equation holds clearly.

Now we consider the case that $\mathbf{F}_w$ is nonempty. Pick a point $x \in \mathbf{F}_w$. We can assume

$$p^{-1}(x) \cap \mathbf{F}_{\tilde{w}} = \{\tilde{x}_1, \dots, \tilde{x}_n\}$$

where $n > 0$ since the covering $p : \widetilde{M} \to M$ is finite and $p(\mathbf{F}_{\tilde{w}}) = \mathbf{F}_w$. If $y \neq x$ and $y \in \mathbf{F}_w$, then there is a Nielsen path c from x to y such that $c \simeq f \circ c$ rel. $\{x, y\}$ by the definition of fixed point class. Hence there are n liftings $\tilde{c}_i$ with $\tilde{c}_i(0) = \tilde{x}_i$ and $\tilde{f} \circ \tilde{c}_i \simeq \tilde{c}_i$ rel. endpoints, $i = 1, \dots, n$. Therefor

$$\{\tilde{c}_1(1), \dots, \tilde{c}_n(1)\} = p^{-1}(y) \cap \mathbf{F}_{\tilde{w}}.$$

Then

$$\#\mathbf{F}_{\tilde{w}} = n\#\mathbf{F}_w.$$

So the equality holds by the fact $\mathrm{ind}(\tilde{f}, \tilde{x}) = \mathrm{ind}(f, x)$. □

A covering $p : \widetilde{M} \to M$ between compact manifolds is called *characteristic* if the subgroup $p_*\pi_1(\widetilde{M})$ is a characteristic subgroup of $\pi_1(M)$, i.e., it is invariant under any automorphism of $\pi_1(M)$. Recall that if G is a finite index subgroup of a finitely generated group H, then there is a finite index characteristic subgroup G' of H, such that $G' \subseteq G$. It follows that given any finite covering $p : \widetilde{M} \to M$ of a compact manifold M, there is a finite covering $q : \widehat{M} \to \widetilde{M}$ so that $p \circ q : \widehat{M} \to M$ is a characteristic covering. In this case for any homeomorphism $f : M \to M$ with

$y = f(x)$, and any points $\tilde{x}$ and $\tilde{y}$ covering x and y respectively, there is a lifting $\tilde{f}$ of f such that $\tilde{f}(\tilde{x}) = \tilde{y}$. So we have the following

Lemma 2.5. *Let $p : \widetilde{M} \to M$ be a characteristic covering between compact manifolds, and $\tilde{w}$ be a lifting of an f-route w of a homeomorphism $f : M \to M$. Then there is a lifting $\tilde{f}$ of f such that $\tilde{w}$ is an $\tilde{f}$-route.*

3. Some facts on Fuchsian groups

In this section, we give some facts on Fuchsian groups.

Let B be an orbifold. Recall that an orbifold in this paper means a compact 2-orbifold with singular points consisting of cone points, then we can assume $B = F(n_1, n_2, \ldots, n_k)$, where the compact surface F denotes the underlying space of B and $n_i \geq 2$ denotes the cone point with cone angle $2\pi/n_i$, $i = 1, 2, \ldots, k$. The fundament group $\pi_1(B)$ is called a *Fuchsian group*. If H is a subgroup of the Fuchsian group $\pi_1(B)$, then H is also a Fuchsian group because there is a covering orbifold $\widetilde{B}$ of B such that $H \cong \pi_1(\widetilde{B})$. Moreover, if $\widetilde{B}$ is also compact, then $\chi(\widetilde{B}) = k\chi(B)$ and H has finite index k in $\pi_1(B)$, where $k > 0$ is the degree of the covering. In particular, if the orbifold B is hyperbolic, then B can be covered by a hyperbolic surface S, and $\pi_1(B)$ is an infinite Fuchsian group since $\pi_1(B)$ has a subgroup isomorphic to the infinite group $\pi_1(S)$. For more information on Fuchsian groups, see [JS, Chapter 2].

Lemma 3.1. (1) *Any infinite Fuchsian group with nontrivial center is either free abelian of rank ≤ 2 or isomorphic to the fundamental group of a Klein bottle.*

(2) *Any finite Fuchsian group is either cyclic or isomorphic to*

$$\langle a, b | a^{n_1} = b^{n_2} = (ab)^{n_3} = 1\rangle,\ \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1, n_1, n_2, n_3 \geq 2,$$

which is the fundamental group of the closed orbifold $O = S^2(n_1, n_2, n_3)$ with $\chi(O) = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 1 > 0$.

Proof. Conclusion (1) is from [JS, Proposition II.3.11], and Conclusion (2) can be verified by [He, Theorem 12.2] clearly. □

For a group G and an element $g \in G$, let $C_G(g) = \{x \in G | xg = gx\}$ be the centralize of g in G, and $C(G) = \{x \in G | xg = gx, \forall g \in G\}$ the center of G. We have

Lemma 3.2. *Let B be a hyperbolic orbifold and $a \in \pi_1(B)$ be an element of infinite order. Then for any $i \neq 0$,*

$$C_{\pi_1(B)}(a^i) = C_{\pi_1(B)}(a) \cong \mathbb{Z}.$$

Proof. For any $i \neq 0$, $C_{\pi_1(B)}(a^i) \leq \pi_1(B)$ is an infinite Fuchsian group with nontrivial center since the infinite cyclic group $\langle a \rangle \leq C_{\pi_1(B)}(a^i)$. Then $C_{\pi_1(B)}(a^i)$ is either free abelian of rank ≤ 2 or the fundamental group of a Klein bottle by Lemma 3.1(1). Note that B is hyperbolic and $C_{\pi_1(B)}(a^i)$ is a subgroup of $\pi_1(B)$, then $C_{\pi_1(B)}(a^i)$ is neither the fundamental group of a Klein bottle nor the fundamental group of a torus because neither of them is hyperbolic. Therefore, $C_{\pi_1(B)}(a^i)$ is free cyclic, set $C_{\pi_1(B)}(a^i) = \langle c \rangle \cong \mathbb{Z}$. Since $a \in C_{\pi_1(B)}(a^i) = \langle c \rangle$, a is a power of c. Thus $\langle c \rangle \leq C_{\pi_1(B)}(a)$, i.e., $C_{\pi_1(B)}(a^i) \leq C_{\pi_1(B)}(a)$. Clearly, $C_{\pi_1(B)}(a) \leq C_{\pi_1(B)}(a^i)$. Thus $C_{\pi_1(B)}(a^i) = C_{\pi_1(B)}(a) \cong \mathbb{Z}$ for any $i \neq 0$. □

Now we give two lemmas which are used in the following.

In group theory, a group G is called *metacyclic* if it contains a cyclic, normal subgroup N such that the quotient group G/N is also cyclic. Clearly, the rank of a metacyclic group is no more than 2. In particular, cyclic groups are metacyclic. For metacyclic groups, we have

Lemma 3.3. (1) *Any subgroup of a metacyclic group is also metacyclic.*

(2) *Let G be a group. If G has a cyclic normal subgroup T such that the quotient group G/T is metacyclic, then any subgroup of G has rank ≤ 3.*

Proof. (1) Let G be a metacyclic group, i.e., there is a cyclic normal subgroup N such that G/N is also cyclic. Let H be any subgroup of G. Then $H \cap N$ is a cyclic normal subgroup of H, and $H/H \cap N \cong HN/N$ is also cyclic since $HN/N \leq G/N$. Thus H is metacyclic.

(2) Let H be any subgroup of G. Note that $H \cap T$ is a cyclic normal subgroup of H because T is cyclic normal in G. Then the quotient group $H/H \cap T \cong HT/T \leq G/T$ is metacyclic and $\mathrm{rank}(H/H \cap T) \leq 2$ according to conclusion (1). Thus $\mathrm{rank}H \leq 3$. □

Lemma 3.4. *Suppose G is a group and $c \in G$ is an element of infinite order. If all the infinite-order elements are contained in the infinite cyclic group $\langle c\rangle$, then G is a metacyclic group. More precisely, either $G = \langle c\rangle \cong \mathbb{Z}$ or $G = \langle a, c | a^2 = 1, aca^{-1} = c^{-1}\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$.*

Proof. If there are no nontrivial finite-order elements in G, then $G = \langle c\rangle \cong \mathbb{Z}$ clearly.

Now we assume that there is an element $a \in G$ of finite order $n \geq 2$. Consider the subgroup $H = \langle a, c\rangle \leq G$. Clearly aca^{-1} is also an element of infinite order, thus $aca^{-1} \in \langle c\rangle$ and $\langle c\rangle$ is a normal subgroup of H. Hence $aca^{-1} = c^i$ for some $i \neq 0$, and H is a metacyclic group. Then

$$c = a^n ca^{-n} = a^{n-1}(aca^{-1})a^{1-n} = a^{n-1}c^i a^{-n+1} = c^{i^n}.$$

Note that c is of infinite order, we have $i^n = 1$. Thus $i = 1$, if n is odd; $i = \pm 1$, if n is even. Namely, H is either

(1) $\langle a, c | a^n = 1, aca^{-1} = c\rangle$, or

(2) $\langle a, c | a^n = 1, aca^{-1} = c^{-1}\rangle, n$ even.

However, case (1) is impossible. In fact, note that $ac \in H$ is of infinite order, thus $ac \in \langle c\rangle$. Then $a \in \langle c\rangle$ which contradicts to the assumption that $a \in G$ has finite order $n \geq 2$.

In case (2), note that $a^2ca^{-2} = c$, we have an abelian subgroup $\langle a^2, c | a^n = 1, a^2ca^{-2} = c\rangle$. It implies a^2c has infinite order. Thus $a^2c \in \langle c\rangle$ and $a^2 \in \langle c\rangle$. Recall that a has finite order $n \geq 2$ and c has infinite order, then $n = 2$. It implies all the nontrivial finite-order elements in G must have order 2, and

$$H = \langle a, c | a^2 = 1, aca^{-1} = c^{-1}\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

To prove $G = H$, it suffices to prove $G \leq H$. Suppose there is another element $b \in G$ of order 2, then there is a subgroup $H' = \langle b, c | b^2 = 1, bcb^{-1} = c^{-1}\rangle$ according to the argument above. Thus $abcb^{-1}a^{-1} = c$, i.e., ab commutes with c. If abc has finite order $k > 0$, then $(ab)^k c^k = (abc)^k = 1$, i.e., $(ab)^k = c^{-k}$ has infinite order. Thus ab has infinite order and $ab \in \langle c\rangle$, it implies $b \in \langle a, c\rangle = H$. If abc has infinite order, then $abc \in \langle c\rangle$ which also implies $b \in \langle a, c\rangle = H$. Therefore, $G \leq H$ and the proof is finished. □

Proposition 3.5. *Let H be a subgroup of $\pi_1(B)$ where B is a hyperbolic orbifold. If*

$$H^d := \{h^d | h \in H\} \subseteq \langle a\rangle \cong \mathbb{Z}$$

where $d \in \mathbb{Z}_+$ is a positive integer and $a \in \pi_1(B)$ of infinite order. Then H is a metacyclic group, more precisely, H is either cyclic or isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.

Proof. Note that B is a hyperbolic orbifold, namely, it can be covered by a hyperbolic surface S. Then $\pi_1(B)$ is an infinite Fuchsian group since it has a subgroup isomorphic to the infinite group $\pi_1(S)$.

If $H^d = \{1\}$, then H consists of elements of finite order. Thus H is a finite Fuchsian group by the fact that every infinite Fuchsian group has an element of infinite order (see [JS, Lemma II.3.9]). Therefore, H is either finite cyclic or isomorphic to the fundamental group $\pi_1(O)$ of a closed orbifold

O with $\chi(O) > 0$ by Conclusion (2) of Lemma 3.1. But the latter is impossible because B with $\chi(B) < 0$ can not be covered by O. Thus H is finite cyclic.

If $H^d \neq \{1\}$, then H contains some elements of infinite order. For any $h \in H$ of infinite order, there is an element $a^{i_h} \neq 1$ such that $h^d = a^{i_h}$ since $H^d \subseteq \langle a \rangle$. Then $h \in C_{\pi_1(B)}(a^{i_h})$. By Lemma 3.2, $C_{\pi_1(B)}(a^{i_h}) = C_{\pi_1(B)}(a) = \langle c \rangle$ for an infinite-order element c, thus $h \in \langle c \rangle$. Then all the infinite-order elements of H are contained in the infinite cyclic group $H \cap \langle c \rangle$. Therefore, H is either infinite cyclic or isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$ by Lemma 3.4, and the proof is finished. □

4. Homeomorphisms of circle bundles over surfaces

In this section, let us consider the special case that M is an orientable circle bundle over an orientable hyperbolic surface S with fibration $p : M \to S$.

Lemma 4.1. *Suppose M is a compact orientable circle bundle over a compact orientable hyperbolic surface S. Then all fibers can be coherently oriented so that they present the same element of infinite order in the center of $\pi_1(M)$.*

Proof. This is clear from the presentation of $\pi_1(M)$ given in [He, Chapter 12]. □

Let t be the element of $\pi_1(M)$ represented by the fiber. We call the infinite cyclic subgroup $\langle t \rangle$ the *fiber* of $\pi_1(M)$ associated to the fibration p, and f *preserves* (resp. *reveres*) *the fiber orientation* if $f_\pi(t) = t$ (resp. $f_\pi(t) = -t$).

Let S be a compact connected hyperbolic surface. A standard form of homeomorphisms on S is developed in [JG] with fine-tuned local behavior from the Thurston canonical map from [T].

Theorem T. *Let S be a compact connected hyperbolic surface. Every homeomorphism $f : S \to S$ is isotopic to a homeomorphism φ such that either*

(1) *φ is a periodic map, i.e., $\varphi^m = id$ for some $m \geq 1$, or equivalently, φ is an isometry with respect to some hyperbolic metric on S; or*

(2) *φ is a pseudo-Anosov map, i.e., there is a number $\lambda > 1$ and a pair of transverse measured foliations $(\mathfrak{F}^s, \mu^s)$ and $(\mathfrak{F}^u, \mu^u)$ such that $\varphi(\mathfrak{F}^s, \mu^s) = (\mathfrak{F}^s, \frac{1}{\lambda}\mu^s)$ and $\varphi(\mathfrak{F}^u, \mu^u) = (\mathfrak{F}^u, \lambda\mu^u)$; or*

(3) *φ is a reducible map, i.e. there is a system of disjoint simple closed curves $\Gamma = \{\Gamma_1, \cdots, \Gamma_k\}$ in $\mathrm{int}S$ with the property below.*

(a) *Γ is invariant by φ (but the Γ_i's may be permuted) and each component of $S \backslash \Gamma$ has negative Euler characteristic.*

(b) *Γ has a φ-invariant tubular neighborhood $\mathcal{N}(\Gamma)$ such that on each (not necessarily connected) φ-component of $S \backslash \mathcal{N}(\Gamma)$, φ satisfies* (1) *or* (2).

(c) *Γ is minimal among all systems satisfying* (a) *and* (b).

(d) *φ is in the* standard form *as defined in* [JG, page 79].

The φ above will be called the *standard form* isotopic to S.

Lemma 4.2. *Let $f : S \to S$ be a homeomorphism of an orientable compact hyperbolic surface S and $\mathbf{F}_w$ be a fixed point class corresponding to an f-route w. Then*

(1) *If $\mathbf{F}_w$ is inessential, then* $\mathrm{rankFix}(f_w) \leq 1$*;*

(2) *If f is orientation-reversing, then* $\mathrm{rankFix}(f_w) \leq 1$.

Proof. Case (1) is clearly following from [JWZ, Theorem 1.1].

Now we consider case (2). Since the index and rank of fixed point classes are both homotopy invariants, via an isotopy, we may assume that f is in standard form. If the fixed point class $\mathbf{F}_w$ is empty, then case (2) holds according to case (1). If $\mathbf{F}_w$ is nonempty, it is connected since each

Nielsen path of S can be deformed (rel. endpoints) into Fix$f$ by [JG, Lemmas 1.2, 2.2 and 3.4]. Thus rankFix$(f_w) = \text{rank}\pi_1(\mathbf{F}_w)$. Then case (2) can be proved by examining the list given in [JG, Lemma 3.6]. □

The lemma below is useful in the proof of Proposition 4.4.

Lemma 4.3. *Suppose $p : M \to S$ is a compact orientable circle bundle over a compact orientable hyperbolic surface S. Suppose $f : M \to M$ is a fiber-preserving homeomorphism that reverse the fiber orientation, and $f' : S \to S$ is the induced homeomorphism. Let w be an f-route. Then the f-fixed point class $\mathbf{F}_w$ is essential if and only if the f'-fixed point class $\mathbf{F}_{p\circ w}$ is essential.*

Proof. Since the index of fixed point classes is a homotopy invariant, via a fiber-preserving homotopy, we can assume that the induced homeomorphism f' is in standard form by [JW, Lemma 2.7]. Then $\text{ind}(f, \mathbf{F}_w)$ equals either $\text{ind}(f', \mathbf{F}_{p\circ w})$ or $2\text{ind}(f', \mathbf{F}_{p\circ w})$ by [JW, Lemma 2.9]. Thus the conclusion holds. □

Proposition 4.4. *Suppose $p : M \to S$ is a compact orientable circle bundle over an orientable hyperbolic surface S, $f : M \to M$ is a fiber-preserving homeomorphism that reveres the orientation of M, and $f' : S \to S$ is the induced homeomorphism of f. Let w be an f-route corresponding to an inessential fixed point class $\mathbf{F}_w$. Then*

(1) $\text{Fix}(f'_{p\circ w})$ *is trivial or the free cyclic group $\mathbb{Z}$;*
(2) $\text{Fix}(f_w)$ *is trivial or a free abelian group of rank ≤ 2.*

Proof. (1) Note that all fibers can be coherently oriented since M is a connected orientable circle bundle over an orientable surface by Lemma 4.1. So f is in one of the two cases below.

Case (i). f is fiber orientation-reversing and $f' : S \to S$ is orientation-preserving.

Since the f-fixed point class $\mathbf{F}_w$ is inessential, the f'-fixed point class $\mathbf{F}_{p\circ w}$ is also inessential by Lemma 4.3. Then $\text{rankFix}(f'_{p\circ w}) \leq 1$ by Lemma 4.2(1).

Case (ii). f is fiber orientation-preserving and $f' : S \to S$ is orientation-reversing. Then $\text{rankFix}(f'_{p\circ w}) \leq 1$ by Lemma 4.2(2).

Hence conclusion (1) holds in both case (i) and case (ii) by the fact that every nontrivial element of an orientable surface group has infinite order.

(2) Let $x = w(0)$. Since $p \circ f = f' \circ p : M \to S$, there is the following commutative diagram on fundamental groups:

$$\begin{array}{ccc} \pi_1(M,x) & \xrightarrow{f_w} & \pi_1(M,x) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ \pi_1(S,p(x)) & \xrightarrow{f'_{p\circ w}} & \pi_1(S,p(x)) \end{array}$$

where

$$p_* : \pi_1(M,x) \to \pi_1(S,p(x)) \cong \pi_1(M,x)/\langle t\rangle$$

is the quotient map and $\langle t\rangle$ is the fiber of $\pi_1(M,x)$ associated with the fibration p. Hence

$$p_*\text{Fix}(f_w) \leq \text{Fix}(f'_{p\circ w}).$$

Since $\langle t\rangle$ is in the center of $\pi_1(M,x)$, and $\text{Fix}(f'_{p\circ w})$ is trivial or the infinite cyclic group $\mathbb{Z}$ by conclusion (1), we have

$$\text{Fix}(f_w) \leq p_*^{-1}\text{Fix}(f'_{p\circ w}) \cong \text{Fix}(f'_{p\circ w}) \times \langle t\rangle,$$

which is isomorphic to the free abelian group $\mathbb{Z}$ or $\mathbb{Z}\oplus\mathbb{Z}$. Hence conclusion (2) holds by the fact that every nontrivial subgroup of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$ is a free abelian group of rank ≤ 2. □

5. Proof of Theorem 1.4

In this section, we give the proof of Theorem 1.4.

Let M be a compact orientable Seifert manifold with hyperbolic orbifold B_M, and $p : M \to B_M$ be the Seifert fibration. If τ is a regular fiber in this fibration, then

$$\langle t\rangle = \mathrm{Im}(\pi_1(\tau, x) \to \pi_1(M, x))$$

is a infinite cyclic subgroup. We call $\langle t\rangle$ the *fiber* of $\pi_1(M, x)$ (associated with the fibration p). In fact, for each Seifert fibration of M, there exists a unique fiber of $\pi_1(M, x)$. It is well known that the cyclic subgroup $\langle t\rangle$ is normal in $\pi_1(M, x)$ and is center when M and B_M are both orientable. The Seifert fibration p induced a quotient homomorphism $p_* : \pi_1(M, x) \to \pi_1(M, x)/\langle t\rangle \cong \pi_1(B_M)$ on fundamental groups (see [JS, Chapter II, §4] or [He, Chapter 12]).

Proposition 5.1. *Let M be a compact orientable Seifert manifold with hyperbolic orbifold B_M, and $f : M \to M$ be an orientation-reversing homeomorphism. Suppose $f_w : \pi_1(M, x) \to \pi_1(M, x)$ is the automorphism induced by f, where w is an f-route with $x = w(0)$. If the fixed point class $\mathbf{F}_w$ corresponding to w is inessential, then* $\mathrm{rankFix}(f_w) \leq 3$.

To prove Proposition 5.1, we need the following useful lemma which is from [JWW, Theorem 3.11].

Lemma 5.2. *Suppose M is a compact orientable Seifert manifold and $p : M \to B_M$ is a Seifert fibration with hyperbolic orbifold B_M. Then any homeomorphism on M is isotopic to a fiber-preserving homeomorphism with respect to this fibration.*

Proof of Proposition 5.1. By Lemma 5.2, f can be isotopic to a fiber-preserving homeomorphism. Since the index $\mathrm{ind}(f, \mathbf{F}_w)$ and rank $\mathrm{rankFix}(f_w)$ of fixed point class are homotopy invariants (see Fact (homotopy invariance) in Section 2), we can assume that f is fiber-preserving and the base point x is in a regular fiber τ in the following.

Note that B_M is hyperbolic, then there is a finite covering $q : S \to B_M$ of orbifold such that S is a compact orientable hyperbilic surface by [S, Theorem 2.5]. The pull-back of the Seifert fibration $p : M \to B_M$ via q gives a finite covering manifold $q' : \widetilde{M} \to M$ with fibration $p' : \widetilde{M} \to B_{\widetilde{M}} = S$. After passing to further finite covering if necessary, we may assume that q' is a characteristic covering with finite sheets d. Since S is an orientable surface, the fibration p' is an orientable circle bundle over the orientable surface S. So there is the following commutative diagram:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "q'"] \arrow[d, "p'"'] & M \arrow[d, "p"] \\ S \arrow[r, "q"] & B_M \end{tikzcd}$$

Pick a point $\tilde{x} \in q'^{-1}(x)$ and a lifting $\tilde{w}$ of w with $\tilde{w}(0) = \tilde{x}$. Since q' is a finite characteristic covering, following from Lemma 2.5, there is a lifting $\tilde{f}$ of f such that $\tilde{w}$ is an $\tilde{f}$-route. Let $\tilde{f}' : S \to S$ denote the induced homeomorphism of $\tilde{f}$ on the orbifold S. Therefore, there is the following

commutative diagram on fundamental groups:

$$\begin{tikzcd}
& \pi_1(S,p'(x)) \arrow[rr, "\tilde{f}'_{p'\circ\tilde{w}}"] \arrow[dd, "q_*"' near end] & & \pi_1(S,p'(x)) \arrow[dd, "q_*"'] \\
\pi_1(\widetilde{M},\tilde{x}) \arrow[ur, "p'_*"] \arrow[rr, "\tilde{f}_{\tilde{w}}" near end] \arrow[dd, "q'_*"'] & & \pi_1(\widetilde{M},\tilde{x}) \arrow[ur, "p'_*"] \arrow[dd, "q'_*"' near start] & \\
& \pi_1(B_M) \arrow[rr] & & \pi_1(B_M) \\
\pi_1(M,x) \arrow[ur, "p_*"] \arrow[rr, "f_w"] & & \pi_1(M,x) \arrow[ur, "p_*"] &
\end{tikzcd}$$

Note that $f : M \to M$ is a fiber-preserving homeomorphism that reverse the orientation of M, then the lifting $\tilde{f} : \widetilde{M} \to \widetilde{M}$ is a fiber-preserving homeomorphism that reverse the orientation of $\widetilde{M}$. Moreover, since the f-fixed point class $\mathbf{F}_w$ is inessential, the $\tilde{f}$-fixed point class $\mathbf{F}_{\tilde{w}}$ is also inessential by Lemma 2.4. Therefore, $\mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}})$ is trivial or the free cyclic group $\mathbb{Z}$ according to conclusion (1) of Proposition 4.4. We claim that

Claim 5.3. *Let $H = p_*\mathrm{Fix}(f_w) \subseteq \pi_1(B_M)$ and $H^d = \{h^d | h \in H\}$. Then*

$$H^d \subseteq q_*\mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}}) \leq \pi_1(B_M)$$

where $q_*\mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}})$ is trivial or the free cyclic group $\mathbb{Z}$.

Proof. Let $\alpha \in \mathrm{Fix}(f_w) \leq \pi_1(M,x)$. Recall that $q' : \widetilde{M} \to M$ is a characteristic covering of sheets d, then

$$q'_* : \pi_1(\widetilde{M},\tilde{x}) \to \pi_1(M,x)$$

is an injective homomorphism and the image $q'_*\pi_1(\widetilde{M},\tilde{x})$ is a characteristic subgroup of $\pi_1(M,x)$ with index d. Hence the power

$$\alpha^d \in q'_*\pi_1(\widetilde{M},\tilde{x}) \leq \pi_1(M,x),$$

furthermore, following from the commutative diagram, we have

$$\tilde{f}_{\tilde{w}}q'^{-1}_*(\alpha^d) = q'^{-1}_*f_w(\alpha^d) = q'^{-1}_*(\alpha^d),$$

and

$$\tilde{f}'_{p'\circ\tilde{w}}p'_*(q'^{-1}_*(\alpha^d)) = p'_*\tilde{f}_{\tilde{w}}(q'^{-1}_*(\alpha^d)) = p'_*q'^{-1}_*(\alpha^d).$$

Namely,

$$p'_*q'^{-1}_*(\alpha^d) \in \mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}}).$$

Then

$$(p_*(\alpha))^d = p_*(\alpha^d) = q_*p'_*q'^{-1}_*(\alpha^d) \in q_*\mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}}) \leq \pi_1(B_M).$$

Hence

$$H^d \subseteq q_*\mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}}) \leq \pi_1(B_M).$$

Note that q_* is induced by the covering q, then q_* is an injective homomorphism, and $q_*\mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}})$ is isomorphic to $\mathrm{Fix}(\tilde{f}'_{p'\circ\tilde{w}})$ which is trivial or the free cyclic group $\mathbb{Z}$. Hence the Claim holds.

Now we will finish the proof of Proposition 5.1.

By Proposition 3.5 and Claim 5.3, $H = p_*\mathrm{Fix}(f_w)$ is a metacyclic group. Since

$$p_* : \pi_1(M,x) \to \pi_1(M,x)/\langle t\rangle = \pi_1(B_M)$$

is the quotient map, we have $\mathrm{Fix}(f_w) \leq p_*^{-1}(H)$ which is an extension of the metacyclic group H by the infinite cyclic group $\langle t\rangle$, namely, $\mathrm{Fix}(f_w)$ is a subgroup of the group $p_*^{-1}(H)$ which has a cyclic normal subgroup $\langle t\rangle$ such that the quotient group H is metacyclic. So by Lemma 3.3, $\mathrm{rankFix}(f_w) \leq 3$. □

Now we give the proof of Theorem 1.4. Firstly, we have the following lemma on the ranks of Seifert manifold groups.

Lemma 5.4. *Suppose M is a compact orientable Seifert manifold with hyperbolic orbifold B_M. Then* $\mathrm{rank}\pi_1(M) \geq 2$.

Proof. It is clear from [Z1, Proposition 4.3] which gives a description on the ranks of Seifert manifold groups. □

Proof of Theorem 1.4. Let $f_\pi = f_w : \pi_1(M, x) \to \pi_1(M, x)$ be induced by some f-route w with $x = w(0)$.

If $\mathbf{F}_w$ is essential, then $\mathrm{rankFix}(f_w) < 2\mathrm{rank}\pi_1(M)$ by Proposition 1.7.

If $\mathbf{F}_w$ is inessential, then $\mathrm{rankFix}(f_w) \leq 3$ by Proposition 5.1. Recall that the orbifold B_M of M is hyperbolic, then $\mathrm{rank}\pi_1(M) \geq 2$ according to Lemma 5.4. Therefore, we have

$$\mathrm{rankFix}(f_w) < 2\mathrm{rank}\pi_1(M).$$

The proof is finished. □

6. Examples and Questuins

Now we give some examples and questions.

Example 6.1. Let S_n be a closed orientable surface of genus $n \geq 2$. Define an orientation-reversing homeomorphism f as follows:

$$f = f_1 \times f_2 : \ S_n \times S^1 \to S_n \times S^1,$$

where $f_1 : S_n \to S_n$ is a reflection on a simple closed curve γ, and $f_2 : S^1 \to S^1$ is a rotation. Then all the fixed point classes of f are inessential, and f induces an automorphism f_π of $\pi_1(S_n \times S^1)$ such that

$$\mathrm{Fix}(f_\pi) = \pi_1(\gamma \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Namely, there is an inessential fixed point class which has $\mathrm{rankFix}(f_\pi) = 2$.

Question 6.2. Is there an orientation-reversing homeomorphism f of a Seifert manifold M whose inessential fixed point class has $\mathrm{rankFix}(f_\pi) = 3$? Namely, is the bound 3 in Proposition 5.1 sharp?

References

[BH] M. Bestvina and M. Handel, *Train tracks and automorphisms of free groups*, Annals of Mathematics, 135 (1992), 1–51.

[He] J. Hempel, *3-manifolds*, Annals of Math. Studies 86, Princeton University Press, 1976.

[J] B. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemporary Mathematics vol. 14, American Mathematical Society, Providence (1983).

[JG] B. Jiang and J. Guo, *Fixed points of surface diffeomorphisms*, Pacific Journal of Mathematics, 160 No.1 (1993), 67–89.

[JS] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs of the American Mathematical Society; V.21, No.220, Providence (1979).

[JW] B. Jiang and S. Wang, *Lefschetz numbers and Nielsen numbers for homeomorphisms on aspherical manifolds*, Topology Hawaii, 1990, World Sci. Publ., River Edge, NJ, 1992, 119–136.

[JWW] B. Jiang, S. Wang and Y. Wu, *3-manifolds and the realization of Nielsen number*, Comm. Anal. Geom. Vol. 9(2001), NO.4, 825–877.

[JWZ] B. Jiang, S. Wang and Q. Zhang, *Bounds for fixed points and fixed subgroups on surfaces and graphs*, Alg. Geom. Topology, 11(2011), 2297–2318.

[LW] J. Lin and S. Wang, *Fixed subgroups of automorphisms of hyperbolic 3-manifold groups*, Topology Appl., 173(2014), 209–218 .

[S] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), 401–487.

[T] W.P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc, 19 (1988), 417–431.

[WZ] J. Wu and Q. Zhang, *The group fixed by a family of endomorphisms of a surface group*, J. Algebra, 417(2014), 412–432 (2014)

[Z1] Q. Zhang, *Bounds for fixed points on Seifert manifolds*, Topology and its Applications, 159(15) (2012), 3263–3273.

[Z2] Q. Zhang, *Bounds for fixed points on hyperbolic 3-manifolds*, Topology Appl., 164(2014), 182–189.

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